

## FINITE DIFFERENCE APPROXIMATION FOR PARABOLIC INTERFACE PROBLEM WITH TIME-DEPENDENT COEFFICIENTS

Bratislav V. Sredojević and Dejan R. Bojović

**ABSTRACT.** The convergence of difference scheme for two-dimensional initial-boundary value problem for the heat equation with concentrated capacity and time-dependent coefficients of the space derivatives, is considered. An estimate of the rate of convergence in a special discrete  $\widetilde{W}_2^{1,1/2}$  Sobolev norm, compatible with the smoothness of the coefficients and solution, is proved.

### 1. Introduction

The finite-difference method is one of the basic tools for the numerical solution of partial differential equations. In the case of problems with discontinuous coefficients and concentrated factors (Dirac delta functions, free boundaries, etc.) the solution has a weak global regularity and it is impossible to establish convergence of finite difference schemes using the classical Taylor series expansion. Often, the Bramble–Hilbert lemma takes the role of the Taylor formula for functions from the Sobolev spaces [6, 8, 12].

Following Lazarov et al. [12], a convergence rate estimate of the form

$$\|u - v\|_{W_{2,h}^k} \leq Ch^{s-k} \|u\|_{W_2^s}, \quad s > k,$$

is called *compatible* with the smoothness (regularity) of the solution  $u$  of the boundary-value problem. Here  $v$  is the solution of the discrete problem,  $h$  is the spatial mesh step,  $W_2^s$  and  $W_{2,h}^k$  are Sobolev spaces of functions with continuous and discrete argument, respectively,  $C$  is a constant which doesn't depend on  $u$  and  $h$ . For the parabolic case typical estimates are of the form

$$\|u - v\|_{W_{2,h\tau}^{k,k/2}} \leq C(h + \sqrt{\tau})^{s-k} \|u\|_{W_2^{s,s/2}}, \quad s > k,$$

---

2010 *Mathematics Subject Classification*: 65M12, 65M15.

*Key words and phrases*: interface problem, convergence, Sobolev norm.

Partially supported the Serbian Ministry of Education, Science and Technological Development (Project 174002).

Communicated by Boško Jovanović.

where  $\tau$  is the time step. In the case of equations with variable coefficients the constant  $C$  in the error bounds depends on norms of the coefficients (see, for example, [1, 8, 18]).

One interesting class of parabolic problems model processes in heat-conducting media with concentrated capacity in which the heat capacity coefficient contains a Dirac delta function, or equivalently, the jump of the heat flow in the singular point is proportional to the time-derivative of the temperature [14]. Such problems are nonstandard and the classical tools of the theory of finite difference schemes are difficult to apply to their convergence analysis.

In the present paper a finite-difference scheme, approximating the two-dimensional initial-boundary value problem for the heat equation with concentrated capacity and time dependent coefficients is derived. Special Sobolev norm (corresponding to the norm  $W_2^{1,1/2}$  for a classical heat-conduction problem) is constructed. In this norm, a convergence rate estimate, compatible with the smoothness of the solution of the boundary value problem, is obtained.

Note that the convergence to classical solutions is studied in [5, 19]. One-dimensional parabolic problem with weak solution is studied in [2, 3, 10]; 2D parabolic problem with variable coefficients (but not time-dependent) is considered in [3, 11].

## 2. Preliminary results

Let  $H$  be a real separable Hilbert space endowed with inner product  $(\cdot, \cdot)$  and norm  $\|\cdot\|$  and  $S$ -unbounded self-adjoint positive definite linear operator, with domain  $D(S)$  dense in  $H$ . It is easy to see that the product  $(u, v)_S = (Su, v)$ ,  $u, v \in D(S)$  satisfies the inner product axioms. The closure of  $D(S)$  in the norm  $\|u\|_S = (u, u)_S^{1/2}$  is a Hilbert space  $H_S \subset H$ . The inner product  $(u, v)$  continuously extends to  $H_S^* \times H_S$ , where  $H_S^* = H_S^{-1}$  is the dual space for  $H_S$ . Spaces  $H_S$ ,  $H$  and  $H_{S^{-1}}$  represent a Gelfand triple  $H_S \subset H \subset H_{S^{-1}}$  with continuous imbeddings. Operator  $S$  extends to the map  $S : H_S \mapsto H_S^*$ . There exist unbounded self-adjoint positive definite linear operator  $S^{1/2}$ , such that  $D(S^{1/2}) = H_S$  and  $(u, v)_S = (Su, v) = (S^{1/2}u, S^{1/2}v)$ . We also define Sobolev spaces  $W_2^s(a, b; H)$ ,  $W_2^0(a, b; H) = L_2(a, b; H)$ , of the functions  $u = u(t)$  mapping the interval  $(a, b) \subset \mathbb{R}$  into  $H$  (see [13, 20]).

Let  $A$  and  $B$  be unbounded self-adjoint positive definite linear operators,  $A = A(t)$ ,  $B \neq B(t)$ , in Hilbert space  $H$ , in general noncommutative, with  $D(A)$  dense in  $H$  and  $H_A \subset H_B$ . We consider the following abstract Cauchy problems:

$$(2.1) \quad B \frac{du}{dt} + Au = f(t), \quad 0 < t < T, \quad u(0) = 0,$$

$$(2.2) \quad B \frac{du}{dt} + Au = \frac{dg}{dt}, \quad 0 < t < T, \quad u(0) = 0,$$

where  $f(t)$  and  $g(t)$  are given and  $u(t)$  is the unknown function with values in  $H$ . Let also assume that  $A_0 \leq A(t) \leq kA_0$  where  $k = \text{const} > 1$  and  $A_0 \neq A_0(t)$  is a constant self-adjoint positive definite linear operator in  $H$ . The following propositions are proved in [2].

LEMMA 2.1. *Let  $f \in L_2(0, T; H_{A_0^{-1}})$ . Then the solution to problem (2.1) satisfies a priori estimate*

$$(2.3) \quad \int_0^T \|u(t)\|_{A_0}^2 dt + \int_0^T \int_0^T \frac{\|u(t) - u(t')\|_B^2}{|t - t'|^2} dt dt' \leq C \int_0^T \|f(t)\|_{A_0^{-1}}^2 dt.$$

LEMMA 2.2. *Let  $g \in W_2^{1/2}(0, T; H_{B^{-1}})$ . Then the solution to problem (2.2) satisfies a priori estimate*

$$\begin{aligned} & \int_0^T \|u(t)\|_{A_0}^2 dt + \int_0^T \int_0^T \frac{\|u(t) - u(t')\|_B^2}{|t - t'|^2} dt dt' \\ & \leq C \left[ \int_0^T \int_0^T \frac{\|g(t) - g(t')\|_{B^{-1}}^2}{|t - t'|^2} dt dt' + \int_0^T \left( \frac{1}{t} + \frac{1}{T-t} \right) \|g(t)\|_{B^{-1}}^2 dt \right]. \end{aligned}$$

An analogous result hold for the operator-difference schemes. Let  $H_h$  be a finite-dimensional real Hilbert space with the inner product  $(\cdot, \cdot)_h$  and the norm  $\|\cdot\|_h$ . For a self-adjoint positive linear operator  $S_h$  in  $H_h$ , by  $H_{S_h}$  we denote the space  $H_{S_h} = H_h$  with the inner product  $(v, w)_{S_h} = (S_h v, w)_h$  and the norm  $\|v\|_{S_h} = (S_h v, v)_h^{1/2}$ .

Let  $\omega_\tau$  be a uniform mesh on  $(0, T)$  with the step size  $\tau = T/m$ ,  $\omega_\tau^+ = \omega_\tau \cup \{T\}$  and  $\bar{\omega}_\tau = \omega_\tau \cup \{0, T\}$ . Further, we shall use standard notation from the theory of the difference schemes [17]. In particular, we set

$$v_{\bar{t}} = v_{\bar{t}}(t) = \frac{v(t) - v(t - \tau)}{\tau}.$$

We consider the implicit operator-difference scheme

$$(2.4) \quad B_h v_{\bar{t}} + A_h v = \varphi(t), \quad t \in \omega_\tau^+, \quad v(0) = 0,$$

where  $A_h = A_h(t)$  and  $B_h \neq B_h(t)$  are linear positive definite self-adjoint operators in  $H_h$ , in general case noncommutative,  $\varphi(t)$  is given and  $v(t)$  is an unknown function with values in  $H_h$ . Let us also consider the scheme

$$(2.5) \quad B_h v_{\bar{t}} + A_h v = \psi_{\bar{t}}, \quad t \in \omega_\tau^+, \quad v(0) = 0,$$

where  $\psi(t)$  is a given mesh function with values in  $H_h$ . Analogously, as in the previous case, we assume that  $A_{h0} \leq A_h(t) \leq k A_{h0}$  where  $k = \text{const} > 1$  and  $A_{h0} \neq A_{h0}(t)$  is a self-adjoint positive linear operator in  $H_h$ . The following analogs of Lemmas 2.1 and 2.2 are proved in [11].

LEMMA 2.3. *The solution  $v$  of operator-difference scheme (2.4) satisfies a priori estimate*

$$\tau \sum_{t \in \omega_\tau^+} \|v(t)\|_{A_{h0}}^2 + \tau^2 \sum_{t \in \bar{\omega}_\tau} \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|v(t) - v(t')\|_{B_h}^2}{|t - t'|^2} \leq C \tau \sum_{t \in \omega_\tau^+} \|\varphi(t)\|_{A_{h0}^{-1}}^2.$$

LEMMA 2.4. *The solution  $v$  of operator-difference scheme (2.5) satisfies a priori estimate*

$$\tau \sum_{t \in \omega_\tau^+} \|v(t)\|_{A_{h0}}^2 + \tau^2 \sum_{t \in \bar{\omega}_\tau} \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|v(t) - v(t')\|_{B_h}^2}{|t - t'|^2}$$

$$\leq C \left[ \tau^2 \sum_{t \in \bar{\omega}_\tau} \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|\psi(t) - \psi(t')\|_{B_h^{-1}}^2}{|t - t'|^2} + \tau \sum_{t \in \bar{\omega}_\tau} \left( \frac{1}{t + \tau} + \frac{1}{T - t + \tau} \right) \|\psi(t)\|_{B_h^{-1}}^2 \right]$$

### 3. Differential problem and its approximation

Let us consider the 2D initial-boundary-value problem for the heat equation in the presence of a concentrated capacity at the line  $x_2 = \xi$ ,  $0 < \xi < 1$ :

$$(3.1) \quad \begin{aligned} (1 + K\delta_\Sigma(x)) \frac{\partial u}{\partial t} - \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left( a_i(x, t) \frac{\partial u}{\partial x_i} \right) &= f, \text{ on } Q, \\ u &= 0, \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) &= u_0(x), \text{ on } \Omega, \end{aligned}$$

where  $\delta_\Sigma(x) = \delta(x_2 - \xi)$  is the Dirac delta function,  $K = \text{const} > 0$  and  $\Omega = (0, 1)^2$ ,  $Q = \Omega \times (0, T)$ . We shall assume that

$$(3.2) \quad \begin{aligned} a_i &\in W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_1) \cap W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_2), \quad \varepsilon > 0, \\ f &\in W_2^{1+\varepsilon, 1/2+\varepsilon/2}(Q), \\ u_0 &\in W_2^2(\Omega_1) \cap W_2^2(\Omega_2), \end{aligned}$$

$$(3.3) \quad u \in W_2^{3,3/2}(Q_1) \cap W_2^{3,3/2}(Q_2) \cap W_2^{3,3/2}(\Sigma \times (0, T)),$$

where  $\Omega_1 = (0, 1) \times (0, \xi)$ ,  $\Omega_2 = (0, 1) \times (\xi, 1)$ ,  $Q_1 = \Omega_1 \times (0, T)$ ,  $Q_2 = \Omega_2 \times (0, T)$ ,  $\Sigma = \{(x_1, \xi) | x_1 \in (0, 1)\}$ . Note that conditions (3.2) express the minimal smoothness requirements on the data under which the solution  $u$  of (3.1) may belong to the function space stated in (3.3). To guarantee that such  $u$  really exists, we also need some additional compatibility conditions at the corners of  $\Omega$  (see [7]). We also assume that  $0 < c_1 \leq a_i(x, t) \leq c_2$ , on  $Q$ .

Let  $\bar{\omega}_h$ -uniform mesh with step size  $h$  in  $\bar{\Omega}$ ,  $\omega_h = \bar{\omega}_h \cap \Omega$ ,  $\omega_{1h} = \bar{\omega}_h \cap ([0, 1) \times (0, 1))$ ,  $\omega_{2h} = \bar{\omega}_h \cap ((0, 1) \times [0, 1))$ ,  $\sigma_h = \omega_h \cap \Sigma$ . Suppose that  $\xi$  is a rational number. Then one can choose step  $h$  so that  $\sigma_h \neq \emptyset$ . Also we assume that the condition  $c_1 h^2 \leq \tau \leq c_2 h^2$  is satisfied. Define the finite differences in the usual way:

$$v_{\bar{x}_i}(x, t) = \frac{v - v^{-i}}{h}, \quad v_{x_i}(x, t) = \frac{v^{+i} - v}{h},$$

where  $v^{\pm i}(x, t) = v(x \pm e_i h, t)$ ,  $e_1 = (1, 0)$ ,  $e_2 = (0, 1)$ . Problem (3.1) can be approximated on the mesh  $\bar{Q}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$  by the following difference scheme with averaged right-hand side:

$$(3.4) \quad \begin{aligned} (1 + K\delta_{\sigma_h})v_{\bar{t}} + L_h v &= T_1^2 T_2^2 T_t^- f, \text{ on } Q_{h\tau}, \\ v &= 0, \text{ on } \gamma_h \times \omega_\tau^+, \quad v(x, 0) = u_0(x), \text{ on } \omega_h, \end{aligned}$$

where  $L_h v = -\frac{1}{2} \sum_{i=1}^2 ((a_i v_{x_i})_{\bar{x}_i} + (a_i v_{\bar{x}_i})_{x_i})$ ,

$$\delta_{\sigma_h}(x) = \begin{cases} 0, & x \notin \sigma_h \\ 1/h, & x \in \sigma_h \end{cases}$$

is the mesh Dirac function, and  $T_1, T_2, T_t^-$  are Steklov averaging operators defined by

$$\begin{aligned} T_1 f(x_1, x_2) &= T_1^\pm f(x_1 \mp h/2, x_2) = \frac{1}{h} \int_{x_1-h/2}^{x_1+h/2} f(x'_1, x_2) dx'_1, \\ T_2 f(x_1, x_2) &= T_2^\pm f(x_1, x_2 \mp h/2) = \frac{1}{h} \int_{x_2-h/2}^{x_2+h/2} f(x_1, x'_2) dx'_2, \\ T_t^- f(x, t) &= T_t^+ f(x, t - \tau) = \frac{1}{\tau} \int_{t-\tau}^t f(x, t') dt'. \end{aligned}$$

Note that these operators are commutative and transform the derivatives to divided differences, for example:

$$T_i^- \frac{\partial u}{\partial x_i} = u_{\bar{x}_i}, \quad T_i^+ \frac{\partial u}{\partial x_i} = u_{x_i}, \quad T_i^2 \frac{\partial^2 u}{\partial x_i^2} = u_{x_i \bar{x}_i}, \quad T_t^- \frac{\partial u}{\partial t} = u_{\bar{t}}.$$

We also define

$$\begin{aligned} T_2^- f(x_1, x_2) &= \frac{1}{h} \int_{x_2-h}^{x_2} \left(1 + \frac{x'_2 - x_2}{h}\right) f(x_1, x'_2) dx'_2, \\ T_2^+ f(x_1, x_2) &= \frac{1}{h} \int_{x_2}^{x_2+h} \left(1 - \frac{x'_2 - x_2}{h}\right) f(x_1, x'_2) dx'_2. \end{aligned}$$

We define the following discrete norms and seminorms:

$$\begin{aligned} \|v\|_{L_2(Q_{h\tau})}^2 &= \tau \sum_{t \in \omega_\tau^+} \|v(\cdot, t)\|_{L_2(\omega_h)}^2, \quad \|v\|_{L_2(\sigma_h \times \omega_\tau)}^2 = \tau \sum_{t \in \omega_\tau^+} \|v(\cdot, t)\|_{L_2(\sigma_h)}^2 \\ |v|_{L_2(\omega_\tau; W_2^{1/2}(\sigma_h))}^2 &= \tau \sum_{t \in \omega_\tau^+} |v(\cdot, t)|_{W_2^{1/2}(\sigma_h)}^2, \\ |v|_{W_2^{1/2}(\omega_\tau; L_2(\omega_h))}^2 &= \tau \sum_{t \in \bar{\omega}_\tau} \tau \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|v(\cdot, t) - v(\cdot, t')\|_{L_2(\omega_h)}^2}{|t - t'|^2}, \\ |v|_{W_2^{1/2}(\omega_\tau; L_2(\sigma_h))}^2 &= \tau \sum_{t \in \bar{\omega}_\tau} \tau \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|v(\cdot, t) - v(\cdot, t')\|_{L_2(\sigma_h)}^2}{|t - t'|^2}, \\ \|v\|_{\widetilde{W}_2^{1/2}(\omega_\tau; L_2(\omega_h))}^2 &= |v|_{W_2^{1/2}(\omega_\tau; L_2(\omega_h))}^2 + \tau \sum_{t \in \bar{\omega}_\tau} \left( \frac{1}{t + \tau} + \frac{1}{T - t + \tau} \right) \|v(\cdot, t)\|_{L_2(\omega_h)}^2, \\ \|v\|_{\widetilde{W}_2^{1/2}(\omega_\tau; L_2(\sigma_h))}^2 &= |v|_{W_2^{1/2}(\omega_\tau; L_2(\sigma_h))}^2 + \tau \sum_{t \in \bar{\omega}_\tau} \left( \frac{1}{t + \tau} + \frac{1}{T - t + \tau} \right) \|v(\cdot, t)\|_{L_2(\sigma_h)}^2, \\ \|v\|_{\widetilde{W}_2^{1,1/2}(Q_{h\tau})}^2 &= \tau \sum_{t \in \omega_\tau^+} \|v(\cdot, t)\|_{W_2^1(\omega_h)}^2 + |v|_{W_2^{1/2}(\omega_\tau; L_2(\omega_h))}^2 + |v|_{W_2^{1/2}(\omega_\tau; L_2(\sigma_h))}^2. \end{aligned}$$

#### 4. Convergence of the difference scheme

In this section we prove the convergence of difference scheme (3.4) in the  $\widetilde{W}_2^{1,1/2}(Q_{h\tau})$  norm. Let  $u$  be the solution to boundary-value problem (3.1) and

$v$  the solution of difference problem (3.4). The error  $z = u - v$  satisfies the finite difference scheme

$$(4.1) \quad \begin{aligned} (1 + K\delta_{\sigma_h})z_{\bar{t}} + L_h z &= \varphi, \text{ on } \omega_h \times \omega_\tau^+, \\ z &= 0, \text{ on } \gamma_h \times \omega_\tau^+, \quad z(x, 0) = 0, \text{ on } \omega_h. \end{aligned}$$

where

$$\begin{aligned} \varphi &= \sum_{i=1}^2 \eta_{i,\bar{x}_i} + \chi_{bart} + \delta_{\sigma_h} \mu_{\bar{t}}, \\ \chi &= u - T_1^2 T_2^2 u, \\ \eta_i &= T_i^+ T_{3-i}^2 T_t^- \left( a_i \frac{\partial u}{\partial x_i} \right) - \frac{1}{2} (a_i + a_i^{+i}) u_{x_i}, \\ \mu &= Ku - T_1^2 (Ku). \end{aligned}$$

Let us set  $\eta_1 = \tilde{\eta}_1 + \delta_{\sigma_h} \hat{\eta}_1$ ,  $\chi = \tilde{\chi} + \delta_{\sigma_h} \hat{\chi}$ , where

$$\hat{\eta}_1 = \frac{h^2}{6} T_1^+ T_t^- \left( \left[ a_1 \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial a_1}{\partial x_2} \frac{\partial u}{\partial x_1} \right]_\Sigma \right), \quad \hat{\chi} = \frac{h^2}{6} \left[ T_1^2 \frac{\partial u}{\partial x_2} \right]_\Sigma,$$

and  $[u]_\Sigma = u(x_1, \xi + 0, t) - u(x_1, \xi - 0, t)$ .

Using Lemmas 2.3 and 2.4, we directly obtain the following a priori estimate for the solution of difference scheme (4.1):

$$(4.2) \quad \begin{aligned} \|z\|_{\tilde{W}_2^{1,1/2}(Q_{h\tau})} &\leq C \left[ \|\eta_2\|_{L_2(Q_{h\tau})} + \|\tilde{\eta}_1\|_{L_2(Q_{h\tau})} + |\hat{\eta}_1|_{L_2(\omega_\tau; W_2^{1/2}(\sigma_h))} \right. \\ &\quad \left. + \|\tilde{\chi}\|_{\tilde{W}_2^{1/2}(\omega_\tau, L_2(\omega_h))} + \|\hat{\chi}\|_{\tilde{W}_2^{1/2}(\omega_\tau, L_2(\sigma_h))} + \|\mu\|_{\tilde{W}_2^{1/2}(\omega_\tau, L_2(\sigma_h))} \right]. \end{aligned}$$

Therefore, in order to estimate the rate of convergence of difference scheme (3.4), it is sufficient to estimate the right-hand side of inequality (4.2).

We decompose  $\eta_i = \eta_{i1} + \eta_{i2} + \eta_{i3}$ , where

$$\begin{aligned} \eta_{i1} &= T_i^+ T_{3-i}^2 T_t^- \left( a_i \frac{\partial u}{\partial x_i} \right) - (T_i^+ T_{3-i}^2 T_t^- a_i) \left( T_i^+ T_{3-i}^2 T_t^- \frac{\partial u}{\partial x_i} \right), \\ \eta_{i2} &= [T_i^+ T_{3-i}^2 T_t^- a_i - 0, 5(a_i + a_i^{+i})] \left( T_i^+ T_{3-i}^2 T_t^- \frac{\partial u}{\partial x_i} \right), \\ \eta_{i3} &= -0, 5(a_i + a_i^{+i}) \left\{ T_i^+ T_{3-i}^2 T_t^- \frac{\partial u}{\partial x_i} - u_{x_i} \right\}. \end{aligned}$$

The term  $\eta_{i1}$  is a bounded bilinear functional of the argument  $(a_i, u) \in W_4^{1,1/2}(e) \times W_4^{2,1}(e)$ ,

$$e = e(x, t) = \{(x'_1, x'_2, t') : x_i < x'_i < x_i + h, |x'_{3-i} - x_{3-i}| < h, t' \in (t - \tau, t)\},$$

and for  $i = 1$ ,  $x_2 \neq \xi$ . Further,  $\eta_{i1} = 0$  whenever  $a_i$  is a constant or  $u$  is a polynomial of degree one in  $x_1$  or  $x_2$  or a constant. Applying the Bramble–Hilbert lemma [6] we get:

$$(4.3) \quad |\eta_{i1}(x, t)| \leq C |a_i|_{W_4^{1,1/2}(e)} |u|_{W_4^{2,1}(e)}.$$

The term  $\eta_{i2}$  is a bounded bilinear functional of the argument  $(a_i, u) \in W_q^{2,1}(e) \times W_{2q/(q-2)}^{1,1/2}(e)$ ,  $q = 2 + \varepsilon$ . Further,  $\eta_{i2} = 0$  whenever  $a_i$  is a polynomial of degree one in  $x_1$  or  $x_2$  or constant or  $u$  is constant. Applying the Bramble–Hilbert lemma we get the following estimate:

$$(4.4) \quad |\eta_{i2}(x, t)| \leq C |a_i|_{W_q^{2,1}(e)} |u|_{W_{2q/(q-2)}^{1,1/2}(e)}.$$

The term  $\eta_{i3}$  is a bounded bilinear functional of the argument  $(a_i, u) \in \mathbb{C}(\overline{Q_k}) \times W_2^{3,3/2}(e)$ ,  $k = 1, 2$ . Further,  $\eta_{i3} = 0$  whenever  $u$  is a polynomial of degree two in  $x_1$  or  $x_2$  and a polynomial of arbitrary degree in  $t$ . Applying the Bramble–Hilbert lemma, we get the estimate

$$(4.5) \quad |\eta_{i3}(x, t)| \leq C \|a_i\|_{\mathbb{C}(\overline{Q_k})} |u|_{W_2^{3,3/2}(e)}.$$

From estimates (4.3)–(4.5), choosing  $i = 2$ , after summation and using the imbeddings

$$(\ast) \quad \begin{aligned} W_2^{2+\varepsilon, 1+\varepsilon/2} &\subset W_4^{1,1/2}, & W_2^{3,3/2} &\subset W_4^{2,1}, \\ W_2^{2+\varepsilon, 1+\varepsilon/2} &\subset W_q^{2,1}, & W_2^{3,3/2} &\subset W_{2q/(q-2)}^{1,1/2}, \\ W_2^{2+\varepsilon, 1+\varepsilon/2} &\subset \mathbb{C}, & \text{for } q &= 2 + \varepsilon, \end{aligned}$$

we get

$$(4.6) \quad \begin{aligned} \|\eta_2\|_{L_2(Q_{h\tau})} &\leq Ch^2 (\|a_2\|_{W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_1)} \|u\|_{W_2^{3,3/2}(Q_1)} \\ &\quad + \|a_2\|_{W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_2)} \|u\|_{W_2^{3,3/2}(Q_2)}). \end{aligned}$$

Let us estimate the term  $\tilde{\eta}_1$ . At the point  $x \notin \sigma_h$ , we have  $\tilde{\eta}_1 = \eta_1$  and estimates (4.3)–(4.5) are valid. At the point  $x \in \sigma_h$ , we decompose

$$\tilde{\eta}_1 = \sum_{k=1}^3 (\eta_{1,k}^- + \eta_{1,k}^+),$$

where  $\eta_{1,k}^\pm$  are defined at the point  $x_2 = \xi \pm 0$

$$\begin{aligned} \eta_{1,1}^\pm &= T_1^+ T_2^{2\pm} T_t^- \left( a_1 \frac{\partial u}{\partial x_1} \right) - 2(T_1^+ T_2^{2\pm} T_t^- a_1) \left( T_1^+ T_2^{2\pm} T_t^- \frac{\partial u}{\partial x_1} \right) \\ &\quad \pm \frac{h}{6} \left( T_1^+ T_t^- \frac{\partial a_1}{\partial x_2} \right) \left[ 2 \left( T_1^+ T_2^{2\pm} T_t^- \frac{\partial u}{\partial x_1} \right) - \left( T_1^+ T_t^- \frac{\partial u}{\partial x_1} \right) \right] \\ &\quad \pm \frac{h}{6} \left[ \frac{a_1 + a_1^{+1}}{2} \left( T_1^+ T_t^- \frac{\partial^2 u}{\partial x_1 \partial x_2} \right) - \left( T_1^+ T_t^- a_1 \frac{\partial^2 u}{\partial x_1 \partial x_2} \right) \right] \\ &\quad \pm \frac{h}{6} \left[ \left( T_1^+ T_t^- \frac{\partial a_1}{\partial x_2} \right) \left( T_1^+ T_t^- \frac{\partial u}{\partial x_1} \right) - \left( T_1^+ T_t^- \frac{\partial a_1}{\partial x_2} \frac{\partial u}{\partial x_1} \right) \right], \\ \eta_{1,2}^\pm &= \left[ 2(T_1^+ T_2^{2\pm} T_t^- a_1) - \frac{a_1 + a_1^{+1}}{2} \mp \frac{h}{3} \left( T_1^+ T_t^- \frac{\partial a_1}{\partial x_2} \right) \right] \times \left( T_1^+ T_2^{2\pm} T_t^- \frac{\partial u}{\partial x_1} \right), \\ \eta_{1,3}^\pm &= \frac{a_1 + a_1^{+1}}{4} \left[ 2 \left( T_1^+ T_2^{2\pm} T_t^- \frac{\partial u}{\partial x_1} \right) - u_{x_1} \mp \frac{h}{3} \left( T_1^+ T_t^- \frac{\partial^2 u}{\partial x_1 \partial x_2} \right) \right]. \end{aligned}$$

The term  $\eta_{1,1}^\pm$  is a bounded bilinear functional of the argument  $(a_1, u) \in W_4^{1,1/2}(e_1^\pm) \times W_4^{2,1}(e_1^\pm)$  where

$$\begin{aligned} e_1^+ &= (x_1, x_1 + h) \times (\xi, \xi + h) \times (t - \tau, t), \\ e_1^- &= (x_1, x_1 + h) \times (\xi - h, \xi) \times (t - \tau, t). \end{aligned}$$

Further,  $\eta_{1,1}^\pm = 0$  whenever  $a_1$  is a constant or  $u$  is a polynomial of degree one in  $x_1$  or  $x_2$  or a constant. Applying the Bramble–Hilbert lemma we get

$$(4.7) \quad |\eta_{1,1}^\pm(x, t)| \leq C |a_1|_{W_4^{1,1/2}(e_1^\pm)} |u|_{W_4^{2,1}(e_1^\pm)}.$$

The term  $\eta_{1,2}^\pm$  is a bounded bilinear functional of the argument  $(a_1, u) \in W_q^{2,1}(e_1^\pm) \times W_{2q/(q-2)}^{1,1/2}(e_1^\pm)$ ,  $q = 2 + \varepsilon$ . Further,  $\eta_{1,2}^\pm = 0$  whenever  $a_1$  is a polynomial of degree one in  $x_1$  or  $x_2$  or constant or  $u$  is a constant. Applying the Bramble–Hilbert lemma we get the estimate

$$(4.8) \quad |\eta_{1,2}^\pm(x, t)| \leq C |a_1|_{W_q^{2,1}(e_1^\pm)} |u|_{W_{2q/(q-2)}^{1,1/2}(e_1^\pm)}.$$

The term  $\eta_{1,3}^\pm$  is a bounded bilinear functional of the argument  $(a_1, u) \in \mathbb{C}(\overline{Q_k}) \times W_2^{3,3/2}(e_1^\pm)$ ,  $k = 1, 2$ . Further,  $\eta_{1,3}^\pm = 0$  whenever  $u$  is a polynomial of degree two in  $x_1$  or  $x_2$  and polynomial of arbitrary degree in  $t$ . Applying the Bramble–Hilbert lemma we get

$$(4.9) \quad |\eta_{1,3}^\pm(x, t)| \leq C \|a_1\|_{\mathbb{C}(\overline{Q_k})} |u|_{W_2^{3,3/2}(e_1^\pm)}.$$

From estimates (4.3)–(4.5) and (4.7)–(4.9), after summation and using imbeddings (\*), we get

$$(4.10) \quad \begin{aligned} \|\tilde{\eta}_1\|_{L_2(Q_{h\tau})} &\leq Ch^2 (\|a_1\|_{W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_1)} \|u\|_{W_2^{3,3/2}(Q_1)} \\ &\quad + \|a_1\|_{W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_2)} \|u\|_{W_2^{3,3/2}(Q_2)}). \end{aligned}$$

Let us estimate the term  $\hat{\eta}_1$ . For  $\phi \in W_2^{1/2}(\Sigma)$ , the following estimate is valid

$$|T_1^+ \phi|_{W_2^{1/2}(\sigma_h)} \leq C |\phi|_{W_2^{1/2}(\Sigma)} \leq C \|\phi\|_{W_2^1(\Omega_k)}, \quad k = 1, 2,$$

wherefrom

$$|\hat{\eta}_1(\cdot, t)|_{W_2^{1/2}(\sigma_h)} \leq Ch^2 (\|T_t^- \nu(\cdot, t)\|_{W_2^1(\Omega_1)} + \|T_t^- \nu(\cdot, t)\|_{W_2^1(\Omega_2)}),$$

where  $\nu = \nu_1 + \nu_2$ , and

$$\nu_1 = a_1 \frac{\partial^2 u}{\partial x_1 \partial x_2}, \quad \nu_2 = \frac{\partial a_1}{\partial x_2} \frac{\partial u}{\partial x_1}.$$

After summation, we have

$$(4.11) \quad \|\hat{\eta}_1\|_{L_2(\omega_\tau, W_2^{1/2}(\sigma_h))} \leq Ch^2 (\|\nu\|_{W_2^{1,0}(Q_1)} + \|\nu\|_{W_2^{1,0}(Q_2)}).$$

Using the Hölder inequality and the imbeddings (\*), we get

$$(4.12) \quad \begin{aligned} \|\nu_1\|_{W_2^{1,0}(Q_k)} &\leq \|a_1\|_{W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_k)} \|u\|_{W_2^{3,3/2}(Q_k)}, \\ \|\nu_2\|_{W_2^{1,0}(Q_k)} &\leq \|a_1\|_{W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_k)} \|u\|_{W_2^{3,3/2}(Q_k)}, \end{aligned} \quad k = 1, 2.$$



From (4.11)–(4.12) we obtain

$$(4.13) \quad |\hat{\eta}_1|_{L_2(\omega_\tau, W_2^{1/2}(\sigma_h))} \leq Ch^2 (\|a_1\|_{W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_1)} \|u\|_{W_2^{3,3/2}(Q_1)} + \|a_1\|_{W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_2)} \|u\|_{W_2^{3,3/2}(Q_2)}).$$

The estimates of terms  $\tilde{\chi}$ ,  $\mu$  and  $\hat{\chi}$  are obtained in [11]:

$$(4.14) \quad \|\tilde{\chi}\|_{\tilde{W}_2^{1/2}(\omega_\tau, L_2(\omega_h))} \leq Ch^2 \sqrt{1/h} (\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)}),$$

$$(4.15) \quad \|\mu\|_{\tilde{W}_2^{1/2}(\omega_\tau, L_2(\sigma_h))} \leq Ch^2 \sqrt{1/h} \|u\|_{W_2^{3,3/2}(\Sigma \times (0,T))},$$

$$(4.16) \quad \|\hat{\chi}\|_{\tilde{W}_2^{1/2}(\omega_\tau, L_2(\sigma_h))} \leq Ch^2 \sqrt{1/h} \|u\|_{W_2^{2,1}(\Sigma \times (0,T))}.$$

Finally, from (4.2)–(4.16) we obtain the following result.

**THEOREM 4.1.** *The solution of problem (3.4) converges in  $\tilde{W}_2^{1,1/2}(Q_{h\tau})$  to the solution of differential problem (3.1), provided  $c_1 h^2 \leq \tau \leq c_2 h^2$ . Furthermore,*

$$\begin{aligned} \|u-v\|_{\tilde{W}_2^{1,1/2}(Q_{h\tau})} &\leq Ch^2 (\max_i \|a_i\|_{W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_1)} + \max_i \|a_i\|_{W_2^{2+\varepsilon, 1+\varepsilon/2}(Q_2)} + l(h)) \\ &\quad \times (\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)} + \|u\|_{W_2^{3,3/2}(\Sigma \times (0,T))}), \end{aligned}$$

where  $l(h) = \sqrt{\log 1/h}$ .

**REMARK 4.1.** The previous estimate is “almost” compatible with the smoothness of the coefficients and solution of differential problem (3.1). The compatibility is spoiled only by the term  $l(h)$ , which slowly increases when  $h \rightarrow 0$ .

**REMARK 4.2.** Convergence in  $\tilde{W}_2^{2,1}(Q_{h\tau})$  is proved in [4].

## References

1. D. R. Bojović, *Convergence of finite difference method for parabolic problem with variable operator*, Lect. Notes Comp. Sci. **1988** (2001), 110–116.
2. D. R. Bojović, B. S. Jovanović, *Convergence of finite difference method for the parabolic problem with concentrated capacity and variable operator*, J. Comp. Appl. Math. **189** (2006), 286–303.
3. ———, *Convergence of a finite difference method for solving 2D parabolic interface problems*, J. Comp. Appl. Math. **236** (2012), 3605–3612.
4. D. R. Bojović, B. V. Sredojević, B. S. Jovanović, *Numerical approximation of a two-dimensional parabolic time-dependent problem containing a delta function*, J. Comp. Appl. Math. **259** (2014), 129–137.
5. I. Braianov, *Convergence of a Crank-Nicolson difference scheme for heat equation with interface in the heat flow and concentrated heat capacity*, Lect. Notes Comp. Sci. **1196** (1997), 58–65.
6. J. H. Bramble, S. R. Hilbert, *Bounds for a class of linear functionals with applications to Hermite interpolation*, Numer. Math. **16** (1971) 362–369.
7. P. Grisvard, *Singularities in Boundary Value Problems*, in: Research in Applied Mathematics, **22**, Masson, Paris, 1992.
8. B. S. Jovanović, *Finite Difference Method for Boundary Value Problems with Weak Solutions*, Posebna izdanja Mat. Instituta **16**, Belgrade, 1993.
9. B. S. Jovanović, P. P. Matus, *On the strong stability of operator-difference schemes in time-integral norms*, Comput. Methods Appl. Math. **1** (2001), 72–85.

10. B. S. Jovanović, L. G. Vulkov, *On the convergence of finite difference schemes for the heat equation with concentrated capacity*, Numer. Math. **89**(4) (2001), 715–734.
11. ———, *Finite difference approximation for some interface problems with variable coefficients*, App. Num. Math. **59** (2009), 349–372.
12. R. D. Lazarov, V. L. Makarov, A. A. Samarskii, *Applications of exact difference schemes for construction and studies of difference schemes on generalized solutions*, Math. Sbornik **117** (1982), 469–480, in: Russian.
13. J. L. Lions, E. Magenes, *Non Homogeneous Boundary Value Problems and Applications*, Springer-Verlag, Berlin and New York, 1972.
14. A. V. Lykov, *Heat and Mass Transfer*, Nauka, Moscow, 1989, in Russian.
15. L. A. Oganessian, L. A. Rukhovets, *Variational-Difference Method for Solving Elliptic Equations*, AN Arm. SSR, 1979, in Russian.
16. M. Renardy, R. C. Rogers, *An Introduction to Partial Differential Equations*, Springer-Verlag, Berlin and New York, 1993.
17. A. A. Samarskii, *Theory of Difference Schemes*, Nauka, Moscow, 1989, in Russian; English edition: Pure and Appl. Math. **240** Marcel Dekker, 2001.
18. A. A. Samarskii, R. D. Lazarov, V. L. Makarov, *Difference Schemes for Differential Equations with Generalized Solutions*, Vysshaya Shkola, Moscow, 1987, in Russian.
19. L. G. Vulkov, *Applications of Steklov-type eigenvalues problems to convergence of difference schemes for parabolic and hyperbolic equations with dynamical boundary conditions*, Lect. Notes Comp. Sci. **1196** (1997), 1309–1364.
20. J. Wloka, *Partial Differential Equations*, Cambridge Univ. Press, Cambridge, 1987.

Faculty of Science  
University of Kragujevac  
Kragujevac  
Serbia  
bratislavsredojevic9@gmail.com  
bojovicd@ptt.rs

(Received 06 10 2015)

(Revised 04 01 2016)